CONTINUED FRACTION EXPANSIONS IN DYNAMICAL SYSTEMS: APPLICATION

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Abstract: Continued fractions play a significant role in the study of rotation numbers and their connection with dynamical systems, especially in low-dimensional cases such as interval maps and circle maps. In this paper, we explore how continued fraction expansions can be used to analyze the behavior of orbits under simple dynamical rules. We consider examples of irrational rotation on the circle and examine the connection between the continued fraction of the rotation number and the system's qualitative properties, such as periodicity, quasi-periodicity, and stability. Particular attention is given to the Gauss map and its dynamical interpretation as a generator of continued fraction digits. We also investigate how the depth and complexity of the continued fraction expansion influence the convergence rate of orbit approximations. Numerical simulations and visualizations are provided to support the theoretical analysis and to illustrate the fractal-like structures arising in related systems. This study serves as an introductory step for undergraduate students interested in number theory, dynamical systems, and mathematical modeling.

Introduction

Dynamical systems governed by simple rules often exhibit complex behavior, especially when irrational numbers are involved. A key tool to analyze such systems is the **continued fraction expansion (CFE),** which provides optimal rational approximations to irrational rotation numbers. We bridge number theory (CFEs) and dynamics (orbit behavior) by focusing on:

- The **Gauss map** G(x) = 1/x mod 1, which generates CFE digits.

- Irrational rotations on the circle

- Numerical visualization of orbits and convergence properties.

Theoretical Background

1. Continued Fractions

A continued fraction expansion of $\alpha \in R$ is:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $a_i \in Z^+$. The convergents $\frac{p_n}{q_n}$ are truncations of the CFE and satisfy:

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}.$$

2. Dynamical Interpretation

Gauss Map: The map $G(x) = \frac{1}{x}$. Mod 1 acts as a shift operator on CFE digits. For $x \in (0,1)$, the digits a_n are obtained via $a_n = \left\lfloor \frac{1}{G^{n-1}(x)} \right\rfloor$.

Rotation numbers: For a circle map f(x) = x + α mod 1, the CFE of α determines whether orbits are:

- **Periodic** (if α is rational).
- **Dence** (if α is irrational).

Numerical Experiments

1. Implementation in MATLAB

- **Gauss Map and CFE Generation**:

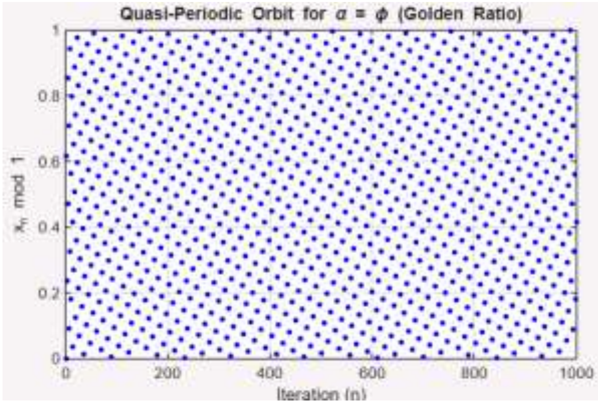
```
function [a, convergents] = gauss_map_cfe(x, n)
   a = zeros(1, n);
   for i = 1:n
      a(i) = floor(1/x);
      x = 1/x - a(i);
   end
end
```

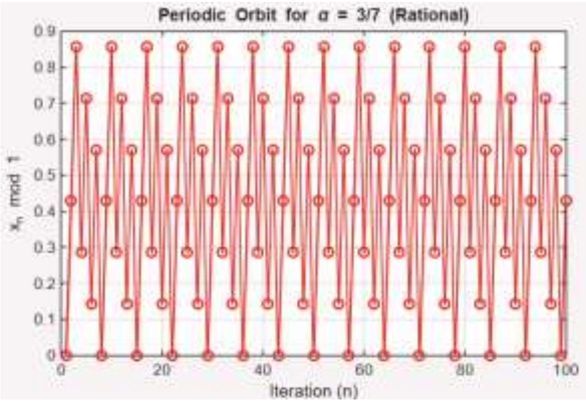
- **Irrational Rotations**

```
alpha = (1 + sqrt(5))/2;
orbit = mod(cumsum([0, alpha*ones(1, 1000)]), 1);
plot(orbit, '.-'); title('Orbit for \alpha = \phi');
```

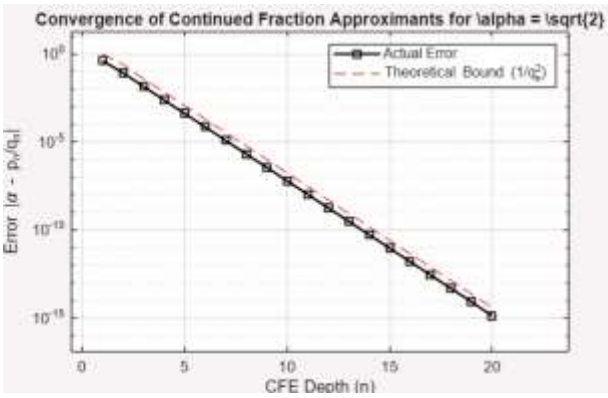
2. Results

- Orbit Visualization:





- Convergence of CFEs:



Discussion

- CFE Depth and Orbit Complexity:

  - Slow convergence for CFEs with large digits (e.g., alpha = [1,1,1,…]).
  - Fractal-like structures in parameter spaces (e.g., Arnold tongues).

- Applications:

- KAM theory (stability of perturbed systems).
- Signal processing (quasi-periodic signals).

Conclusion

We demonstrated how continued fraction expansions elucidate the behavior of dynamical systems, particularly for irrational rotations. Numerical experiments confirmed theoretical predictions, showing how CFE depth correlates with orbit complexity. Future work could extend this to higher-dimensional systems or chaotic maps.

References

1. Bosma, W., Kraaikamp, C., et al. *Continued Fractions*. 2012–2013.
2. Katok, A. *Dynamical Systems with Hyperbolic Behavior*. 1995.
3. MATLAB Documentation. *Numerical Computing*. 2023.