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ECONOMIC PROBLEMS IN LINEAR AND NONLINEAR PROGRAMMING: THE GRAPHICAL METHOD AND THE METHOD OF LAGRANGE MULTIPLIERS

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Abstract: This scientific work examines key economic optimization problems that arise in both linear and nonlinear programming contexts. The graphical method is presented as an intuitive tool for solving two-variable linear programming problems, including resource allocation and production planning. The method of Lagrange multipliers is explored as a powerful technique for nonlinear constrained optimization, with applications to cost minimization and utility maximization in economics. Illustrative numerical examples demonstrate both methods in practical economic settings.

Keywords: linear programming, nonlinear programming, graphical method, Lagrange multipliers, objective function, constraints, optimal solution, resource allocation, cost minimization, utility maximization.

Mathematical programming methods are now irreplaceable tools that help us analyze many complex decision-making and resource allocation problems. Whether an enterprise seeks to maximize profit, minimize production costs, or optimally allocate scarce resources, both linear and nonlinear programming provide reliable frameworks for obtaining optimal decisions. The graphical method and the method of Lagrange multipliers are two classical solution approaches among the most widely applied techniques in economics and management science.

1. The Graphical Method in Linear Programming

Consider the general linear programming problem with two decision variables x_1 and x_2:

Maximize (or Minimize): $Z = c_1x_1 + c_2x_2$

Subject to:

$$a_{11}x_1 + a_{12}x_2 \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 \leq b_2$$

$$x_1, x_2 \geq 0$$

The graphical method proceeds as follows:

1. Plot each constraint as a straight line in the x_1x_2 plane, identifying the feasible region as the intersection of all half-planes satisfying the constraints.
2. Identify the corner (extreme) points of the feasible region, as the optimal solution always occurs at a vertex of this region.
3. Evaluate the objective function Z at each corner point and select the vertex that yields the maximum (or minimum) value.

1.1. Economic Example: Production Planning

A manufacturing firm produces two goods — Product A and Product B. Each unit of A requires 2 machine-hours and 1 labor-hour; each unit of B requires 1 machine-hour and 3 labor-hours. The firm has at most 100 machine-hours and 120 labor-hours available per week. The profit per unit is 50 monetary units for A and 40 for B. The problem is to determine production quantities x_1 (units of A) and x_2 (units of B) that maximize total profit.

Maximize: $Z = 50x_1 + 40x_2$

Subject to:

$$2x_1 + x_2 \leq 100 \quad \text{(machine-hours)} \quad ...(1)$$

$$x_1 + 3x_2 \leq 120 \quad \text{(labor-hours)} \quad ...(2)$$

$$x_1, x_2 \geq 0 \quad ...(3)$$

The corner points of the feasible region are determined by solving the intersections of the constraint boundaries:

Corner Point	x_1	x_2	$Z = 50x_1 + 40x_2$
O	0	0	0
A	50	0	2 500
B	36	28	**2 920 ← Maximum**
C	0	40	1 600

Table 1. Evaluation of the objective function at corner points of the feasible region.

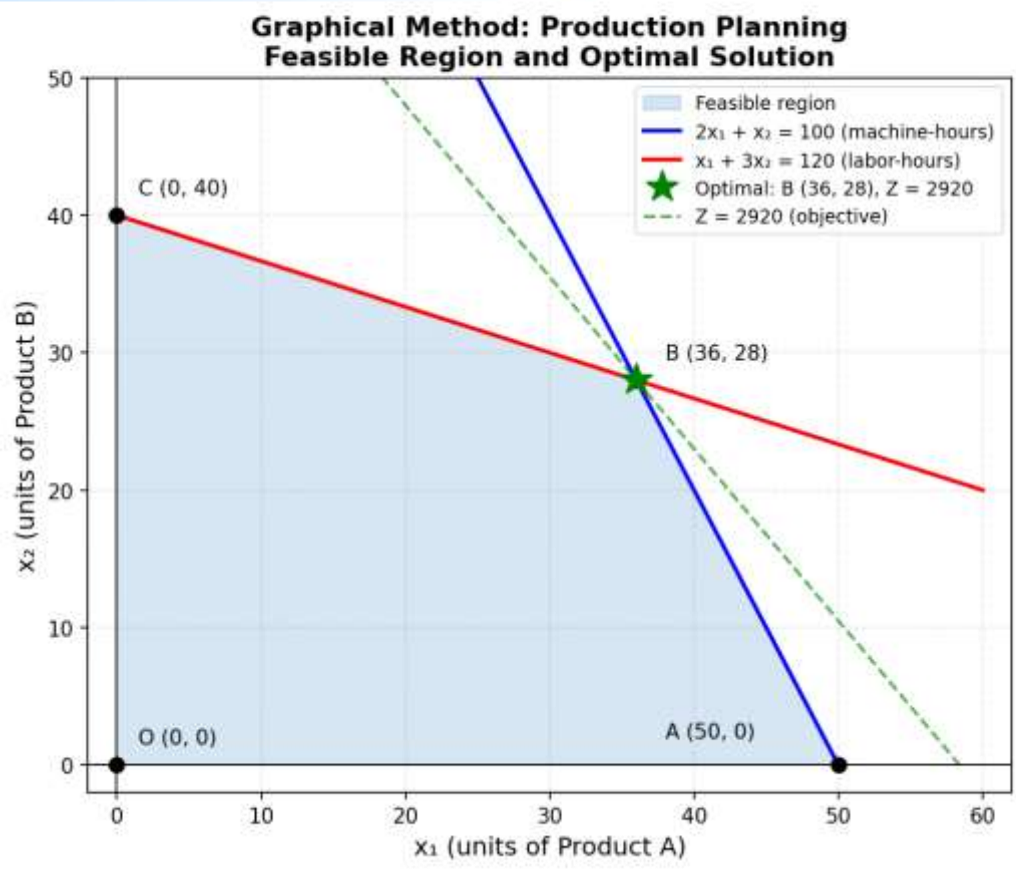

Figure 1. Feasible region and optimal solution for the production planning problem.

The intersection of constraints (1) and (2) is found by solving $2x_1 + x_2 = 100$ and $x_1 + 3x_2 = 120$, yielding $x_1 = 36$, $x_2 = 28$. The maximum profit $Z = 2\,920$ monetary units is achieved at point B ($x_1 = 36$, $x_2 = 28$). The firm should therefore produce 36 units of Product A and 28 units of Product B per week.

2. The Method of Lagrange Multipliers in Nonlinear Programming

When the objective function or constraints are nonlinear, the graphical method is generally inapplicable. The method of Lagrange multipliers provides a systematic analytical technique for solving constrained nonlinear optimization problems. Consider the general form:

Optimize: $f(x_1, x_2, ..., x_n)$

Subject to: $g(x_1, x_2, ..., x_n) = 0$ ...(4)

The Lagrangian function is constructed as:

$L(x_1, ..., x_n, \lambda) = f(x_1, ..., x_n) - \lambda \cdot g(x_1, ..., x_n)$ ...(5)

where λ is the Lagrange multiplier. The necessary conditions for an optimum (first-order conditions) require:

$\partial L/\partial x_i = \partial f/\partial x_i - \lambda(\partial g/\partial x_i) = 0, \quad i = 1, 2, ..., n$ ...(6)

$\partial L/\partial\lambda = -g(x_1, ..., x_n) = 0$ ...(7)

2.1. Economic Example: Consumer Utility Maximization

A consumer seeks to maximize utility $U(x_1, x_2) = x_1^2 \cdot x_2$, where x_1 and x_2 denote quantities of two goods, subject to the budget constraint $p_1x_1 + p_2x_2 = I$. Let prices $p_1 = 2$, $p_2 = 4$, and income $I = 120$.

The constraint is written as:

$$g(x_1, x_2) = 2x_1 + 4x_2 - 120 = 0 \quad ...(8)$$

The Lagrangian is:

$$L = x_1^2 \cdot x_2 - \lambda(2x_1 + 4x_2 - 120) \quad ...(9)$$

Applying first-order conditions:

$$\partial L/\partial x_1 = 2x_1x_2 - 2\lambda = 0 \quad \Rightarrow \quad x_1x_2 = \lambda \quad ...(10)$$

$$\partial L/\partial x_2 = x_1^2 - 4\lambda = 0 \quad \Rightarrow \quad \lambda = x_1^2/4 \quad ...(11)$$

$$\partial L/\partial\lambda = -(2x_1 + 4x_2 - 120) = 0 \quad ...(12)$$

From (10) and (11): $x_1x_2 = x_1^2/4$, giving $x_2 = x_1/4$. Substituting into the budget constraint:

$$2x_1 + 4(x_1/4) = 120 \quad \Rightarrow \quad 3x_1 = 120 \quad \Rightarrow \quad x_1^* = 40$$

$$x_2^* = 40/4 = 10, \quad \lambda^* = 40 \times 10 = 400$$

The optimal consumption bundle is $x_1^* = 40$ and $x_2^* = 10$, yielding maximum utility $U^* = 40^2 \times 10 = 16{,}000$. The Lagrange multiplier $\lambda = 400$ represents the marginal utility of income — each additional monetary unit of budget raises utility by approximately 400 units.

2.2. Economic Example: Cost Minimization

A firm produces output $Q = x_1^{(1/2)} \cdot x_2^{(1/2)}$ (Cobb-Douglas technology) and seeks to minimize total cost $C = w_1x_1 + w_2x_2$ subject to producing a required output level Q_0. With factor prices $w_1 = 3$, $w_2 = 12$ and $Q_0 = 6$, the problem becomes:

Minimize: $C = 3x_1 + 12x_2$

Subject to: $x_1^{(1/2)} \cdot x_2^{(1/2)} = 6$ ...(13)

The Lagrangian is:

$$L = 3x_1 + 12x_2 - \lambda(x_1^{(1/2)} \cdot x_2^{(1/2)} - 6) \quad ...(14)$$

First-order conditions yield:

$$3 = \lambda \cdot (1/2)\, x_1^{(-1/2)}\, x_2^{(1/2)} \quad ...(15)$$

$$12 = \lambda \cdot (1/2)\, x_1^{(1/2)}\, x_2^{(-1/2)} \quad ...(16)$$

Dividing (15) by (16): $x_2/x_1 = 1/4$, so $x_1 = 4x_2$. Substituting into the constraint: $(4x_2)^{(1/2)} \cdot x_2^{(1/2)} = 6$, giving $2x_2 = 6$, so $x_2^* = 3$ and $x_1^* = 12$. Minimum cost: $C^* = 3(12) + 12(3) = 72$ monetary units.

3. Comparative Summary

The table below summarizes the key characteristics of both methods:

Criterion	Graphical Method	Lagrange Multipliers
Type of problem	Linear	Nonlinear
Number of variables	2 (practical limit)	n variables
Constraint type	Inequalities (≤, ≥)	Equalities (=)
Solution approach	Geometric / corner points	Analytical / calculus
Economic meaning of λ	Shadow price of constraint	Marginal value of constraint
Main applications	Production planning, resource allocation	Utility maximization, cost minimization

Table 2. Comparison of the graphical method and the method of Lagrange multipliers.

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