MODELING AND IMPLEMENTATION OF A PROGRAM FOR COMPUTING THE FACTORIAL OF FRACTIONAL NUMBERS USING THE GAMMA FUNCTION

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Abstract. This article provides information about the origin and application of the Gamma function. Formulas for calculating the factorial of fractional numbers using the Gamma function have been derived. A program was developed in the Python programming language to compute the results. The outcomes of manual calculations and the program's results have been compared.

The Gamma function is considered one of the major mathematical objects. The origin of the Gamma function dates back to the 18th century. The Gamma function is associated with the name of Leonhard Euler, as he was the first to introduce the fundamental ideas of the Gamma function in his scientific research. Euler realized the need to generalize the factorial for real numbers and aimed to define the Gamma function. Through this function, he achieved significant results in studying problems such as multi-dimensional integrals, infinite series, and analytic continuation. Later, based on Euler’s work, Legendre made a substantial contribution to shaping the key properties of the Gamma function. For this reason, in some sources, the Gamma function is also referred to as the Euler-Legendre function.

Key words: Gamma function, Legendre, Euler-Legendre function, fractional numbers, the factorial.

Introduction. The Gamma function is an extended form of the ordinary factorial, which works not only for integers but also for fractional numbers.The Gamma function has a wide range of applications in statistics, physics, medicine, and finance. In statistics, the application of the Gamma function is mainly related to probability distributions. Specifically, it is used in calculating models like the Gamma distribution and the Poisson distribution. For example, each patient in a doctor's queue is accepted approximately every 10 minutes. However, the consultation time is not always exact - sometimes it's shorter, sometimes longer. This scenario can be modeled using the Gamma distribution. The Gamma function is used to calculate such uncertain time intervals, i.e., the average waiting time is determined using the Gamma function. It is widely used in building such statistical models.

In physics, the Gamma function is also extensively applied in areas such as energy distribution, heat transfer, and quantum mechanics. For instance, radioactive elements decay over time, but the exact time of this decay is unknown in advance. The time it takes for half of a sample of uranium to decay follows a probability distribution, and the Gamma function is used to determine the decay rate.

Banks and insurance companies use the Gamma function in practice to assess the likelihood of clients repaying loans or the risk of default. For this, probability models based on the Gamma function are used. By analyzing the clients' past payment history, the probability of future repayment is evaluated.

Similarly, the Gamma function is widely applied in medicine as well. It is used to estimate the duration of diseases, the retention time of medications in the body, and the life expectancy of individuals. For instance, after a drug enters the human body, it breaks down over a certain period. However, each person's body is different, so the exact duration of the drug's effectiveness is unknown. Doctors use models based on the Gamma function to determine the average effective duration of a medication.

The Gamma function can be viewed as a kind of continuation of the factorial. As we know, the factorial for positive integers is calculated in a very straightforward manner. The Gamma function is designed to compute the factorial of fractional, negative, and even complex-valued numbers.

Materials and Methods. The Gamma function cannot be expressed through elementary functions, but it can be written in the following integral form:

$$\Gamma(a)=\int_0^\infty e^{-x}\ x^{a-1}\,dx\ .$$

$\Gamma(a)=\int_0^\infty e^{-x}\ x^{a-1}\,dx$, by substituting $a+1$ instead of a in the gamma function $\Gamma(a+1)=\int_0^\infty e^{-x}\ x^{a}\,dx$ we can derive it

$$\Gamma(a+1)=\int_0^\infty e^{-x}\ x^{a}\,dx=\begin{bmatrix}u=x^a;\ \ du=a\cdot x^{a-1}dx\\ dv=e^{-x}dx;\ \ v=-e^{-x}\end{bmatrix}$$

$$\Gamma(a+1)=\int_0^\infty e^{-x}\ x^{a}\,dx\ =(-x^a\cdot e^{-x})\Big|_0^\infty+\int_0^\infty e^{-x}\cdot a\cdot x^{a-1}dx.$$

In the integral above, the expression $x^a \cdot e^{-x}$ tends to zero as $x \to \infty$ because the exponential function grows faster than the polynomial. Additionally, this expression is equal to zero when $x = 0$Therefore, this term completely disappears. From this, we derive the following expression:

$$\Gamma(a+1) = a \cdot \int_0^{\infty} e^{-x} \cdot x^{a-1} dx$$

This expression is the Gamma function itself.

It follows that $\Gamma(a+1) = a \cdot \Gamma(a)$

From this, we can conclude that$\Gamma(a) = (a-1) \cdot \Gamma(a-1)$

When $a \in N$,the expression becomes:

$$\Gamma(a) = (a-1) \cdot (a-2) \cdot (a-3) \cdot ... \cdot 3 \cdot 2 \cdot 1 \cdot \Gamma(1).$$

Now, we calculate $\Gamma(1)$,where $a = 1$ so:

$$\Gamma(1) = \int_0^{\infty} e^{-x}\ x^{1-1}\, dx = \int_0^{\infty} e^{-x}\, dx = -e^{-x} \Big|_0^{\infty} = -(e^{-\infty} - e^{-0}) = 1.$$

If $\Gamma(1)$=1, then the expression above becomes $\Gamma(a) = (a-1) \cdot (a-2) \cdot (a-3) \cdot ... \cdot 3 \cdot 2 \cdot 1$

This can be expressed as $\Gamma(a) = (a-1)!$. From this, we can deduce that $\Gamma(a+1) = a!$.

It is known that $a! = a \cdot (a-1)!$ .Using the formulas $\Gamma(a+1) = a!$ and $\Gamma(a+1) = a \cdot \Gamma(a)$, we deduce that $a! = a \cdot \Gamma(a)$,where $a \in N$.

From this expression, we can calculate the factorial of non-negative integers using the Gamma function, meaning we have linked the Gamma function and the factorial.

From the formula.$\Gamma(a+1)= a!$ by calculating 0!, we find that 0!= $\Gamma(1)$=1 Based on the above formula, we now proceed to calculate the factorial of fractional numbers using the Gamma function. $\Gamma(a)=\int_0^{\infty} e^{-x}\ x^{a-1}\, dx$.

If we assign the value $\frac{1}{2}$ to a, then our expression becomes:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} e^{-x} x^{\frac{1}{2}-1}\, dx = \int_0^{\infty} \frac{dx}{e^x \sqrt{x}} = \begin{bmatrix} \sqrt{x} = u\ ; \\ x = u^2; \\ dx = 2udu \end{bmatrix} = \int_0^{\infty} \frac{2u\, du}{e^{u^2}\, u} = 2\int_0^{\infty} e^{-u^2} du\ .$$

It is known that the Gauss integral over the entire line is expressed as follows:

$$\int_{-\infty}^{\infty} e^{-u^2}\ du = \sqrt{\pi}.$$

Since, $e^{-u^2}$is an even function, the Gauss integral can be expressed as follows:

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = 2\int_0^{\infty} e^{-u^2}\, du.$$

Thus, from the above equation, we derive the expression $2\int_0^{\infty} e^{-u^2}\, du = \sqrt{\pi}$ which leads to the conclusion that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Using the formula $\Gamma(a+1) = a \cdot \Gamma(a)$:

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right); \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right);$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2} \cdot \Gamma\left(\frac{5}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right).$$

Thus, if we write these expressions in general form, we obtain the following expression:

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!!}{2^n} \cdot \Gamma\left(\frac{1}{2}\right) \quad (n \in N).$$

Using the expression $\Gamma(a) = (a-1)!$,we can rewrite the above formula as follows:

$$\left(\frac{2n-1}{2}\right)! = \frac{(2n-1)!!}{2^n} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \cdot \sqrt{\pi} \;;$$

From this, we arrive at the expression $\left(\frac{2n-1}{2}\right)! = \frac{(2n-1)!!}{2^n} \cdot \sqrt{\pi}$.

With this formula, we can also calculate the factorial of fractional numbers.

Let's calculate the factorial of fractional numbers for some values of *n*:

For n=1 $\left(\frac{1}{2}\right)! = \frac{(1)!!}{2^1} \cdot \sqrt{\pi} = \frac{1}{2} \cdot \sqrt{\pi} \approx 0{,}887;$

For n=2 $\left(\frac{3}{2}\right)! = \frac{(3)!!}{2^2} \cdot \sqrt{\pi} = \frac{3 \cdot 1}{4} \cdot \sqrt{\pi} \approx 1{,}33;$

For n=3 $\left(\frac{5}{2}\right)! = \frac{(5)!!}{2^3} \cdot \sqrt{\pi} = \frac{5 \cdot 3 \cdot 1}{8} \cdot \sqrt{\pi} \approx 3{,}32;$

Below is a Python program developed to calculate the factorial of fractional numbers

Using this program, we will compare the calculated values of the factorial of fractional numbers with the values obtained through regular computation.

As can be seen, there is almost no difference between the results produced by the program and the manually calculated values. With the help of this program, we can compute the factorial of fractional numbers of any order.

Gamma Function for Negative Numbers:

For negative numbers, especially negative integers, the gamma function is undefined at certain points (poles), meaning there are singular points. However, if we are talking about negative but non-integer numbers such as $-\frac{1}{2}, -\frac{3}{2}$, and so on, then the gamma function exists and can be calculated. For this, we use Euler's Reflection Formula:

$$\Gamma(\boldsymbol{a}) \cdot \Gamma(\mathbf{1}-\boldsymbol{a}) = \frac{\boldsymbol{\pi}}{\sin(\boldsymbol{\pi a})}$$

If a is a negative but non-integer number, then the gamma function $\Gamma(\boldsymbol{a})$can be expressed using this formula. For example, for $= -\frac{1}{2}$:

1) $\Gamma\left(-\frac{1}{2}\right) \cdot \Gamma\left(\mathbf{1}+\frac{1}{2}\right) = \frac{\pi}{\sin(\pi\cdot(-\frac{1}{2}))}$

$$\Gamma\left(-\frac{1}{2}\right) \cdot \Gamma\left(\frac{3}{2}\right) = -\boldsymbol{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right) \cdot \frac{\sqrt{\pi}}{2} = -\boldsymbol{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right) = -\mathbf{2}\sqrt{\boldsymbol{\pi}}.$$

2) $\Gamma\left(-\frac{3}{2}\right) \cdot \Gamma\left(\mathbf{1}+\frac{3}{2}\right) = \frac{\pi}{\sin\left(\pi\cdot\left(-\frac{3}{2}\right)\right)}$

$$\Gamma\left(-\frac{3}{2}\right) \cdot \Gamma\left(\frac{5}{2}\right) = \boldsymbol{\pi}$$

$$\Gamma\left(-\frac{3}{2}\right) \cdot \frac{3\sqrt{\pi}}{4} = \boldsymbol{\pi}$$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3}$$

$\Gamma(a) = (a-1)!$, and we can also express this expression in the following form:

$$a! = \Gamma(a+1)$$

$$-\frac{1}{2}! = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi};$$

$$-\frac{3}{2}! = \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi};$$

$$-\frac{5}{2}! = \Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3};$$

By using these formulas, we can now calculate the factorial of negative non-integer numbers using the gamma function as well

Conclusions

In conclusion, the gamma function is not just a mathematical function, but it is also effectively used in statistical analysis, physical processes, financial risk assessment, and medical research, as well as in everyday scientific investigations, modeling, and computations

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